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Modeling of seismic wave attenuation in soil structures using fractional derivative scheme

Ünal Dikmen[1]

[1] Cumhuriyet University, Faculty of Engineering, Dept. of Geophysical Eng., 58140 Sivas, Turkey.

Abstract: *The main problem in determination of dynamic behavior of soil structures under dynamic excitations, for instance earthquake loading, is the uncertainty of seismic energy dissipation mechanisms. In application, the damping process is represented by frequency dependent and empirical relations or alternatively a constant damping ratio is used.*

This paper simulates the damping process by a seismic energy dissipation depending upon the strain history in a soil material. This kind of damping scheme is named "fractional damping". Two dimensional soil models have different type of physical, geometrical properties are modeled by the use of finite element method, and the results are compared with the Rayleigh damping procedure.

Key Words**:** *Fractional Order Derivative, Seismic Attenuation, Modeling, Finite Element Method.*

INTRODUCTION

Subsurface structures may focus the seismic energy to a certain area and the seismic energy scattered by near surface heterogeneities may produce surface waves of large amplitudes. Seismic wave propagation through the medium reduces the amplitude of waves. This reduction is the consequence of energy losses in the subsoil and it is named as "attenuation". The attenuation characteristics could consist of unique information about the lithology and dynamic properties of a specific structure. The determination of dynamic properties of soil structures is the fundamental study of seismic design in earthquake engineering.

Seismic wave attenuation in a soil environment is a complex phenomenon resulting from the interaction of several mechanisms that contribute to seismic energy dissipation during dynamic loading. Several definitions have been proposed as measures of seismic energy dissipation in soil materials. These damping schemes are based on the frequency dependent attenuation (Idriss et al., 1973, Hudson et al., 1994), constant damping ratio (Schnabel et al., 1972) or some empirical relations (Hardin and Drnevich 1972a, 1972b). However, most of them are dimensionless and has no physical insight. The frequency dependent relations for damping process have deficiencies especially when process requires long time intervals. Among some frequency dependent relations, the Rayleigh damping scheme is commonly used in practice since it enters to the solution of motion as a linear term and permits uncomplicated numerical calculations. Hardin and Drnevich (1972a, 1972b) pointed out the effect of frequency on damping in soil materials. Additionally, they showed the important role of deformation history by laboratory experiments using a variety of soil specimens. The numerical modeling of the equation of motion may help better understanding of the influence the above-mentioned site conditions on strong ground motion. Analytical solution of motion equation is only possible for simple geometric bodies that exhibit time invariant physical properties. In real geophysical problems, the physical properties of subsurface vary in lateral and vertical directions. For this reason, the equation of motion could only be solved by numerical modeling.

In this study, a new damping approach that performs seismic energy dissipation depending upon strain history in a soil material will proposed by using fractional order derivatives. Various types of two-dimensional (2D) soil models are calculated by means of finite element method and then are compared with Rayleigh damping procedure that depends on frequency (Hudson et al., 1994).

GRÜNWALDIAN DEFINITION OF FRACTIONAL ORDER DERIVATIVE

Several definitions exist to approximate fractional order derivatives. Among some others, the Reimann-Louiville and Grünwald-Letnikov definitions are well-known and can be transformed into each another (Oldham and Spanier, 1974). Grünwald-Letnikov definition is an expansion of the integer order derivative into fractional order. It is preferred because of its easy numerical implementation and it requires less number of restrictions on the type of basis function. Grünwald-Letnikov definition starts with the backward differences approximation of the integer order derivative. Oldham and Spanier (1974) introduced extensive information on fractional order differointegrals equations and their applications to various types of problems in applied sciences. However, a short summary will be given in the following.

The integer order derivative in terms of backward difference leads us to form a general formulation:

$$\frac{d^n f(t)}{dt^n} = \lim_{\Delta t \to \infty}\left[(\Delta t)^{-n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(t - i\Delta t) \right] \tag{1}$$

where n is an integer number. The binomial coefficient in equation (1) is given as follows:

$$\binom{n}{i} = \begin{cases} \dfrac{n!}{i!(n-i)!} & \text{for } 0 \le i \le n \\ 0 & \text{for } 0 \le n \le i \text{ and } i > n. \end{cases} \tag{2}$$

If the time increment Δt is replaced by the Δt=t/N, (N=1, 2, 3…) then equation (1) can be written as

$$\frac{d^n f(t)}{dt^n} = \lim_{\Delta t \to \infty}\left[(\frac{t}{N})^{-n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(t - i\frac{t}{N}) \right], \tag{3}$$

where, *t* and *N* are the time and the number of data point respectively. The upper and lower limits of the sum in equation (3) are called as “terminals” (Oldham and Spanier, 1974). The upper limit can be chosen somewhat arbitrarily. On the other hand, the lower limit has to be zero for taking derivative of a function. To extend equation (3) for any real order derivative, one can use extended definition of the binomial coefficient:

$$\binom{\alpha}{i} = \begin{cases} \dfrac{\alpha(\alpha-1)(\alpha-2)...(\alpha-i-1)}{i} & \text{for i} > 0 \\ 1 & \text{for i} = 0 \end{cases} \tag{4}$$

where α and i are real and integer numbers respectively. Substituting equation (4) into equation (3), the expression for $(-1)^i\binom{n}{i}$ (i>0) can be written as

$$(-1)^i\binom{\alpha}{i}=(-1)^i\,\frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-i+1)}{i!}=\binom{i-\alpha-1}{i} \tag{5}$$

The fractional binomial coefficient can be defined in terms of the well-known Gamma function $\Gamma()$ properties (Abramowitz and Stegun, 1965)

$$\Gamma(n)=(n-1)\Gamma(n-1) \tag{6}$$

that lead to a new form for the equation (5):

$$(-1)^i\binom{\alpha}{i}=\binom{i-\alpha-1}{i}=\frac{\Gamma(i-\alpha)}{\Gamma(-\alpha)\Gamma(i+1)}. \tag{7}$$

Detailed information on Gamma function and its properties may be found in (Oldham and Spanier, 1974; Abramowitz and Stegun, 1965). Substituting equation (7) into equation (3), one can obtain a general formulation for fractional order derivatives:

$$\frac{d^\alpha f(t)}{dt^\alpha}=\lim_{N\to\infty}\left[(\frac{t}{N})^{-\alpha}\sum_{i=0}^{N-1}\frac{\Gamma(i-\alpha)}{\Gamma(-\alpha)\Gamma(i+1)}f(t-i\frac{t}{N})\right] \tag{8}$$

wherein,

$$\frac{\Gamma(i-\alpha)}{\Gamma(-\alpha)\Gamma(i+1)}=A_{i+1}=\frac{i-\alpha-1}{i}A_i \qquad ; \qquad A_1=1 \tag{9}$$

are called "Grünwald coefficients" (Oldham and Spanier, 1974). Grünwald-Letnikov definition (equation 8) for the fractional order derivative is also valid for integer order derivatives and integrals as can be seen from the Riemann sum. An integer order derivative only depends upon the function's local values while the fractional order derivative includes both local and past values of the function. Therefore, the fractional order derivative operator shows global behavior. The history of the function incorporated into its behavior by the use of some weighted coefficients called as "memory effect" (Oldham and Spanier, 1974). When α is a real number, all Grünwald coefficients in equation (9) are different from zero. If α is an integer number, only the first $\alpha+1$ Grünwald coefficients are non zero and set up a local operator. Figure 1 illustrates the variation of Grünwald coefficients (A_{i+1}) versus index i+1 and the order of derivative (α). Table 1 shows some selected values of Grünwald coefficients.

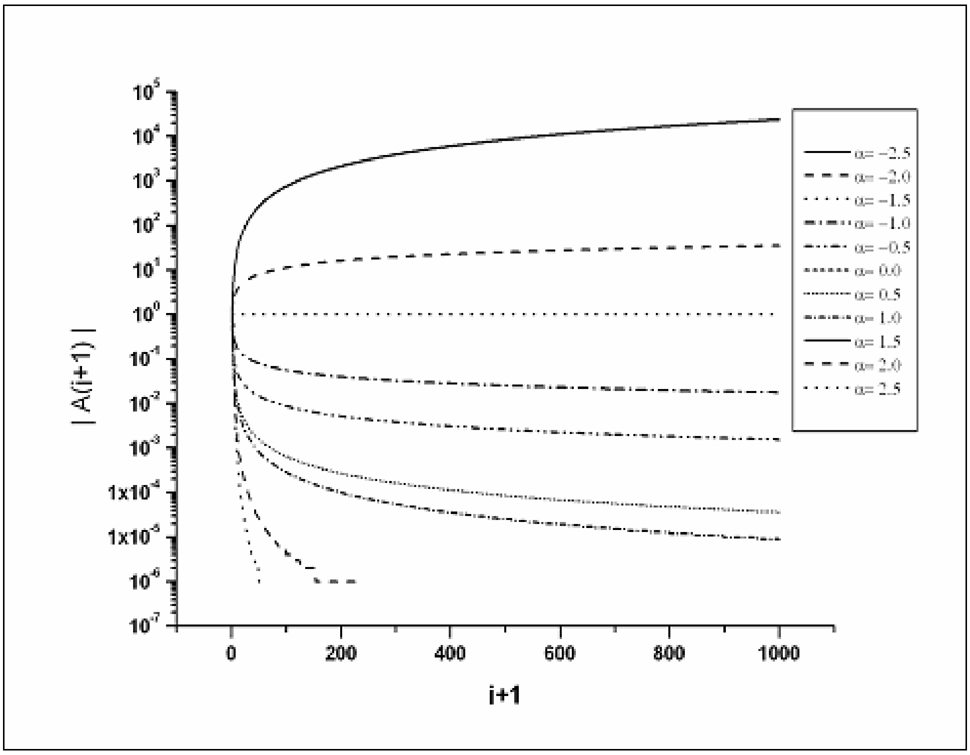

FIG. 1. The variation of Grünwald coefficients | A_{i+1}| versus index i+1 and the order of derivative α.

Table 1.

	α						
J+1	**-1.5**	**-1.0**	**-0.5**	**0.0**	**0.5**	**1.0**	**1.5**
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.0000	0.5000	0.2500	0.0000	0.2500	0.5000	1.0000
3	1.0000	0.3750	0.1562	0.0000	0.0937	0.1250	0.0000
4	1.0000	0.3125	0.1171	0.0000	0.0546	0.0625	0.0000
5	1.0000	0.2734	0.0952	0.0000	0.0375	0.0390	0.0000
10	1.0000	0.1854	0.0525	0.0000	0.0133	0.0109	0.0000
20	1.0000	0.1285	0.0301	0.0000	0.0051	0.0034	0.0000
30	1.0000	0.1043	0.0219	0.0000	0.0030	0.0018	0.0000
40	1.0000	0.0900	0.0176	0.0000	0.0021	0.0011	0.0000
50	1.0000	0.0803	0.0148	0.0000	0.0015	0.0008	0.0000
100	1.0000	0.0566	0.0087	0.0000	0.0006	0.0002	0.0000
200	1.0000	0.0399	0.0052	0.0000	0.0002	0.0001	0.0000
300	1.0000	0.0326	0.0038	0.0000	0.0001	0.0000	0.0000
400	1.0000	0.0282	0.0030	0.0000	0.0001	0.0000	0.0000
500	1.0000	0.0252	0.0026	0.0000	0.0000	0.0000	0.0000
997	1.0000	0.0178	0.0015	0.0000	0.0000	0.0000	0.0000
998	1.0000	0.0178	0.0015	0.0000	0.0000	0.0000	0.0000
999	1.0000	0.0178	0.0015	0.0000	0.0000	0.0000	0.0000
1000	1.0000	0.0178	0.0015	0.0000	0.0000	0.0000	0.0000

FRACTIONAL ORDER CONSTITUTIVE EQUATION FOR VISCOELASTIC MODEL

Damping is defined by the dissipation of mechanical energy produced by some non-conservative forces acting on a material and may be classified into internal and external components. Internal damping is caused by physical phenomena closely linked to the structure of the material, while external damping caused by forces such as Coulomb damping due to dry friction (Ammon, 2001). Especially, the damping could not be ignored in the case of extensive values of stress or strain. Therefore, the Hook's law has to be modified by different constitutive equations by taking account of these effects. Both current and past stress states determine the deformation response of a viscoelastic material. Viscoelastic materials showing such properties are said to have "memory effect" (Koeller, 1987). The fundamental relations for mathematical modeling should also constitute these properties of viscoelastic materials. Some mechanical models consisting of Hookien springs (Figure 2a), and Newtonian dashpot (Figure 2b) that represent elastic and damping forces respectively, is usually used to simulate the damping in a viscoelastic material.

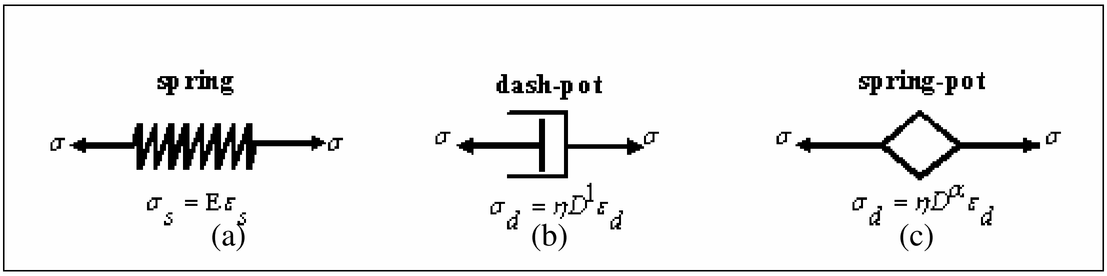

FIG. 2. Mechanical elements. a) Hookien spring, b) dash-pot, c) spring-pot.

In viscoelasticity theory, the stress and strain developed in a material is simply related as

$$(1 + a_1 D^1 + a_2 D^2 + \ldots + a_n D^n)\sigma = (b_0 + b_1 D^1 + b_2 D^2 + \ldots + b_m D^m)\varepsilon \quad (10)$$

(Dillard, 1999). D is the differential operator and a_i, b_i are all positive coefficients that have to be chosen in consistent with the law of hydrodynamics, m and n are the highest order of derivatives of stress and strain respectively (Dillard, 1999). The constitutive equation for a Kelvin-Voigt type viscoelastic model (Figure 3) is given as

$$\sigma = E\varepsilon + \eta \frac{d\varepsilon}{dt}, \quad (11)$$

where E and η represent young modulus and viscosity coefficient, respectively.

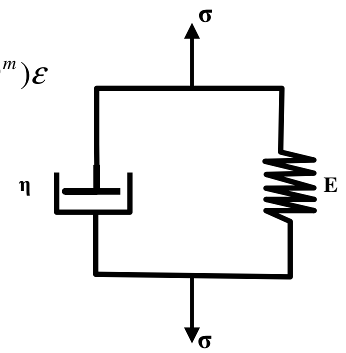

FIG. 3. Kelvin-Voigt type viscoelastic model

Koeller (1987) showed the advantages of fractional order constitutive equation replacing dashpot (Fig. 2b) by spring-pot elements (Fig. 2c) in Kelvin-Voigt type mechanical model to represent better behavior of stress and strain state in the equation (11). Such fractional order mechanical models include spring-pot type damping element depicted in Figure 4.

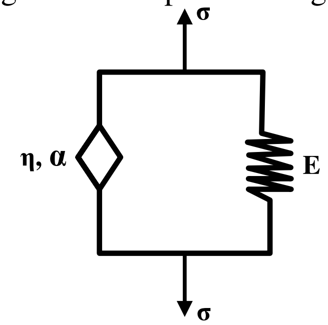

FIG. 4. Fractional Kelvin-Voigt type viscoelastic model.

The equation for such model is given as

$$\sigma = E\varepsilon + \eta \frac{d^{\alpha}\varepsilon}{dt^{\alpha}},$$

(12)

where, α is the fractional order of the derivative. The fractional order Kelvin-Voigt type model is called as "simplest complex model" (Gaul, 1999). Increasing the number of model elements improves damping behavior in mechanical models. However, in such case an extremely complex analysis is required (Gaul, 1999).

FINITE ELEMENT FORMULATION

The fractional order constitutive equations may be combined in order to examine seismic attenuation in soil structures. In the finite element method, the displacement type formulation is given as

$$u = N\hat{u},$$

(13)

where u, N and û are displacement, shape (interpolation) function and displacement at nodal points of finite element. For plane strain condition, strain and nodal displacement are linked by the following equation

$$\varepsilon = B\hat{u},$$

(14)

where, B denoted the appropriate spatial derivatives of shape functions N and called as "Kinematics matrix" (Zeinkiewich and Taylor, 1991). The stress field can be obtained due to nodal displacement by substituting equation (14) into equation (12):

$$\sigma = EB\hat{u} + \eta BD^{\alpha}\hat{u}$$

(15)

where, D denotes derivative operator. The principle of virtual work yields the equation of motion in the form of

$$\int_v \varepsilon^T \sigma\, dv + \int_v F_B u^T\, dv + \int_s f_s\, u^T ds = 0$$

(16)

(Dikmen, 2004). Wherein F_B and F_s are body and external forces and uppercase T denotes the transpose of a matrix. The first and second integral signs in equation (16) denote the region on which the finite element is specified and s defines the predefined surfaces on which external forces act. Substituting equations (14) and (15) into equation (16), the equation of motion for multi degrees of freedom systems takes the following form:

$$mD^2\hat{u}(t) + cD^{\alpha}\hat{u}(t) + k\hat{u}(t) = f(t)$$

(17)

where,

$$k = \int_v B^T EB\, dv$$

(18a)

$$c = \int_v B^T \eta B \, dv \tag{18b}$$

and

$$m = \int_v N^T \rho N \, dv \tag{18c}$$

stiffness, damping and consistent mass matrices respectively and ρ denotes the density of the material. In order to solve the equation of motion (equation 17) for nodal displacement at time *t+1*, Newmark-Beta algorithm gives an unconditional stable solution (Newmark, 1959). Newmark-Beta method takes the form (Bathe, 1984) for displacement and velocity at time (t+Δt)

$$u(t+\Delta t) = u(t) + \Delta t\, \dot{u}(t) + \frac{\Delta t^2}{4}\left[\ddot{u}(t) + \ddot{u}(t+\Delta t)\right] \tag{19}$$

and

$$\dot{u}(t+\Delta t) = \dot{u}(t) + \frac{\Delta t}{2}\left[\ddot{u}(t) + \ddot{u}(t+\Delta t)\right] \tag{20}$$

where, the upper dot symbols denotes derivatives with respect to time. Based on (19), (20) and (8), the equation of motion may be

$$\left[\frac{4}{\Delta t^2} m + \left(\frac{t}{N}\right)^{-\alpha} A_1 c + k\right] \hat{u}(t+\Delta t) = f(t+\Delta t) + m\left[\frac{4}{\Delta t^2}\hat{u}(t) + \frac{4}{\Delta t}\dot{\hat{u}}(t) + \ddot{\hat{u}}(t)\right] - c\left(\frac{t}{N}\right)^{-\alpha} \sum_{i=0}^{N-1} A_{i+1} \hat{u}\left(t - i\frac{t}{N}\right) \tag{21}$$

where the effective stiffness matrix is

$$k_{\text{eff}} = \left[\frac{4}{\Delta t^2} m + \left(\frac{t}{N}\right)^{-\alpha} A_1 c + k\right] \tag{22}$$

and effective force vector is

$$f_{\text{eff}}(t+\Delta t) = f(t+\Delta t) + m\left[\frac{4}{\Delta t^2}\hat{u}(t) + \frac{4}{\Delta t}\dot{\hat{u}}(t) + \ddot{\hat{u}}(t)\right] - c\left(\frac{t}{N}\right)^{-\alpha} \sum_{i=0}^{N-1} A_{i+1} \hat{u}\left(t - i\frac{t}{N}\right). \tag{23}$$

Consequently, a linear equation system is obtained as follow:

$$k_{\text{eff}}\, \hat{u}(t+\Delta t) = f_{\text{eff}}(t+\Delta t)\,. \tag{24}$$

The displacements on nodal points of finite element mesh are calculated numerically by using equation (24).

NUMERICAL EXAMPLE

The example embankment model taken from Hudson et al. (1994) consists of 6000 feet width and 100 feet thickness. Figure 5 shows the finite element mesh applied to the model. The horizontal acceleration and shear stresses on nodes and elements of finite element mesh calculated by using both Quad4m (Hudson et al., 1994) and Dyn2d (Dikmen, 2004) algorithms to compare Rayleigh and fractional damping procedures.

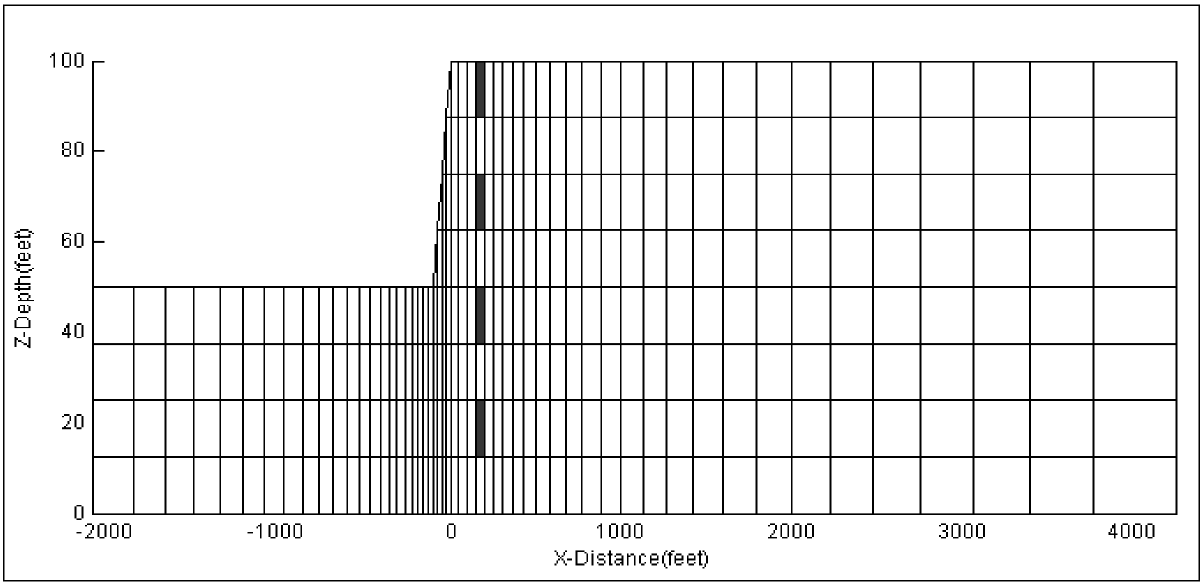

FIG. 5. Finite element mesh of the synthetic model.

Table 2 and 3 illustrate the model input parameters for Quad4m and Dyn2d algorithms, respectively. Loma-Priate 1989, California earthquake acceleration record is used as bedrock input in model calculations.

Figure 6 and 7 shows the earthquake acceleration record and its Fourier spectrum. The horizontal acceleration histories are calculated at the nodes 164 and 167 in both algorithms.

Table 2.

n	Node-1	Node-2	Node-3	Node-4	Type	Dens (pcf)	po	Gmx	G	XL	Str-el
1	1	2	7	6	1	120.0	0.45	345	249.0	0.08	0
.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.
330	378	379	388	387	1	120.0	0.45	345	210.0	0.11	0

Table 3.

n	Node-1	Node-2	Node-3	Node-4	Dens (pcf)	Vp (fps)	Vs (fps)	Str-el
1	1	2	7	6	120.0	1000.0	300.0	0
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
330	378	379	388	387	120.0	1000.0	300.0	0

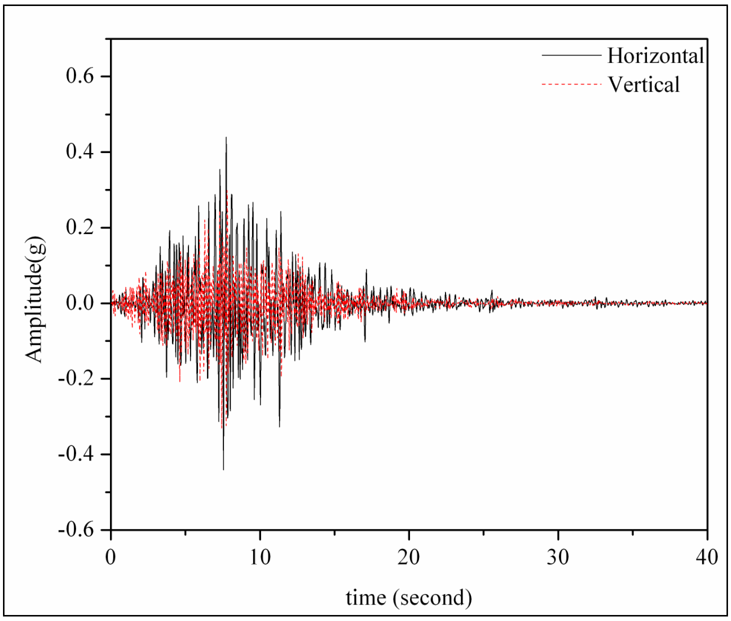

FIG. 6. Earthquake acceleration record used in synthetic model.

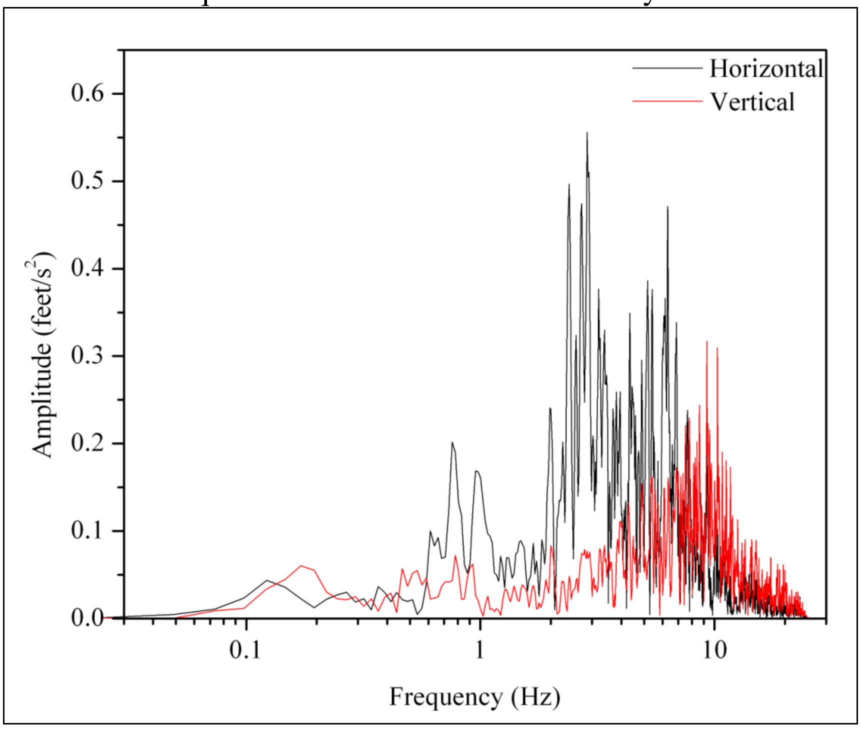

FIG. 7. Fourier spectrum of earthquake acceleration record.

The horizontal acceleration history at node 164 and its Fourier spectrum are shown in Figure 8 and in Figure 9, respectively.

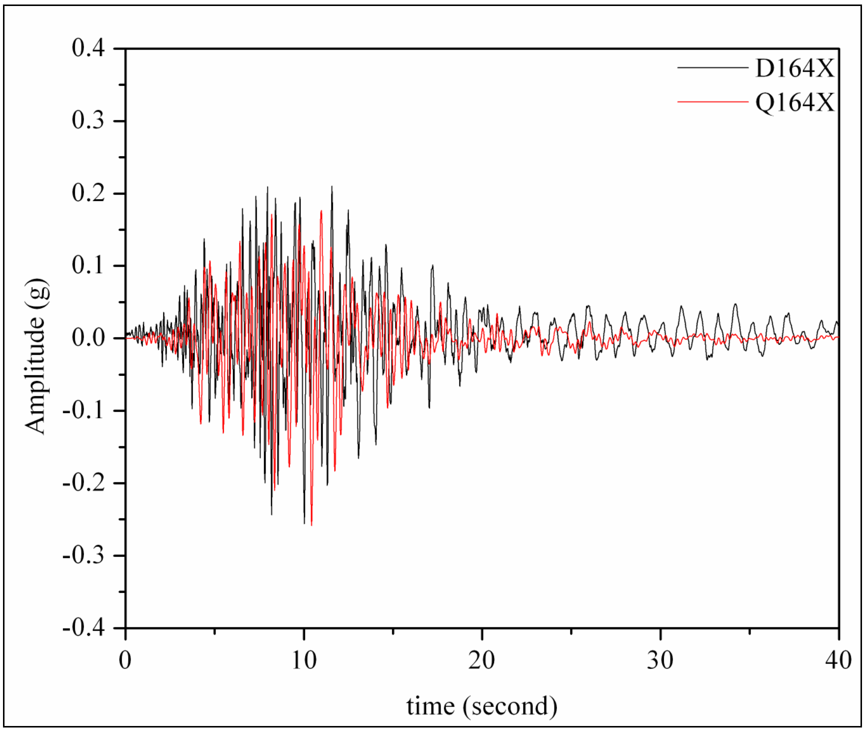

FIG. 8. Calculated horizontal acceleration history at 164th node point.

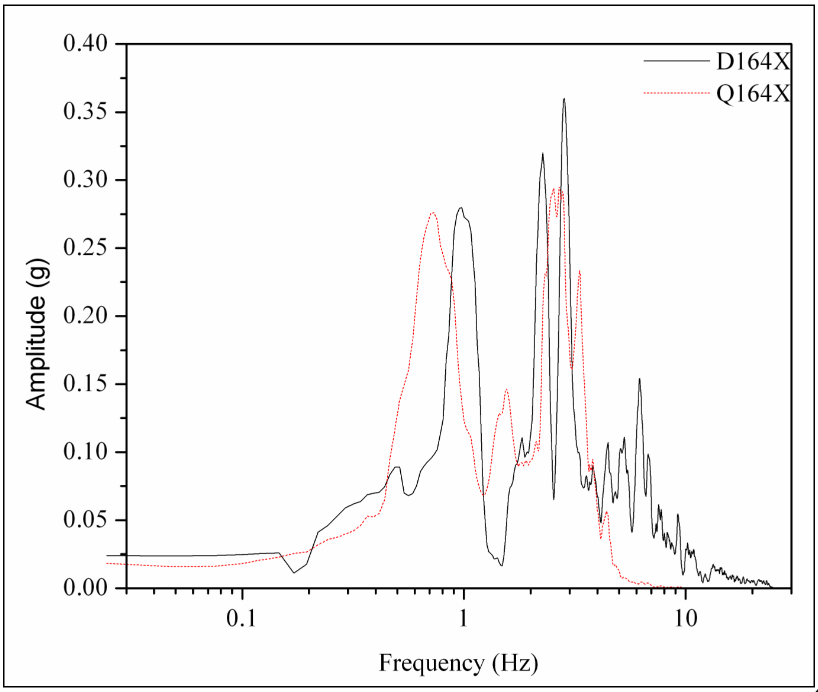

FIG. 9. Fourier spectrum of calculated horizontal acceleration history at 164th node point.

Figure 10 and 11 illustrate the horizontal acceleration history at node 167 and its Fourier spectrum, respectively.

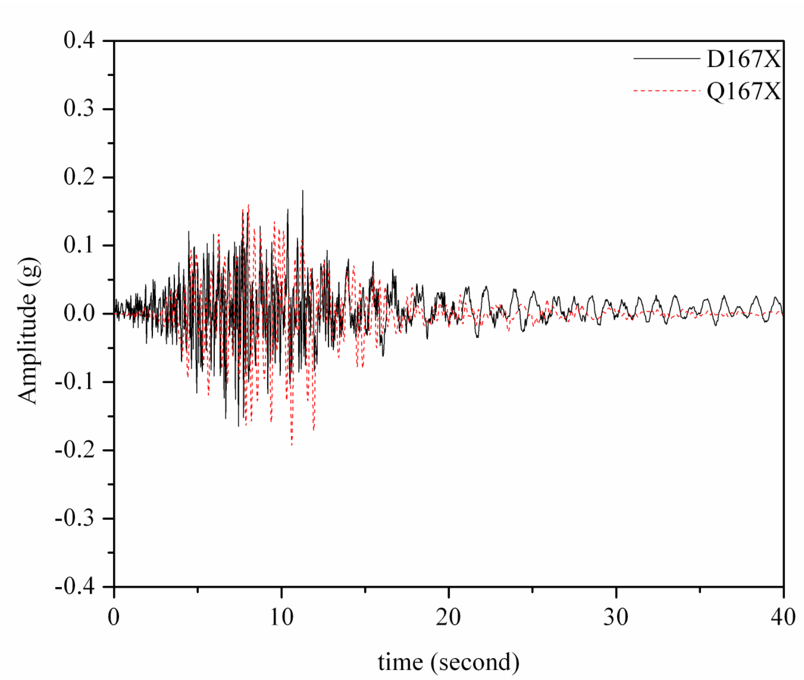

FIG. 10. Calculated horizontal acceleration history at 167th node point.

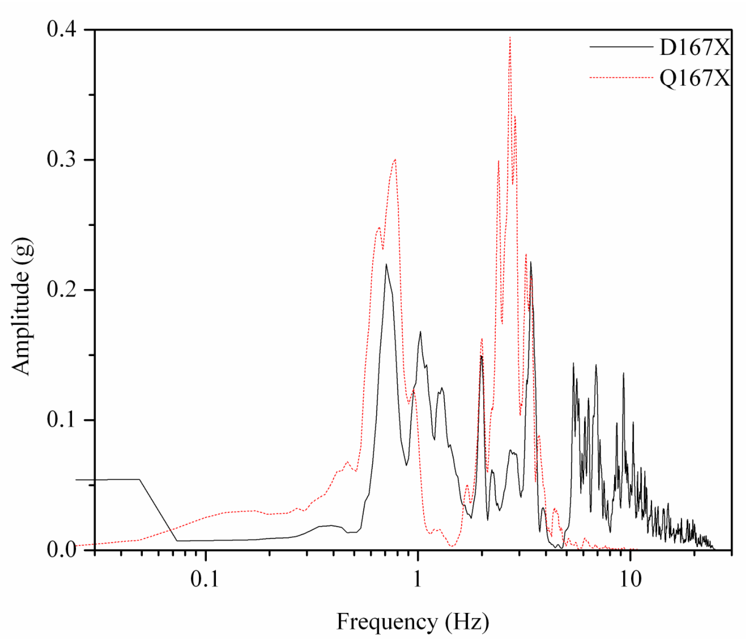

FIG. 11. Fourier spectrum of calculated horizontal acceleration history at 167th node point.

Figure 12, 13, 14 and 15 shows the calculated shear stresses by both algorithm corresponding to the elements 139, 141, 143 and 145 that are indicated by black boxes in Figure 5. Figure 16 shows the variation of shear modulus ratio (G/Gmax) and damping ratio (D/Dmax) calculated by Dyn2d versus shear strain. The variation of maximum shear stress with depth is shown in Figure 17.

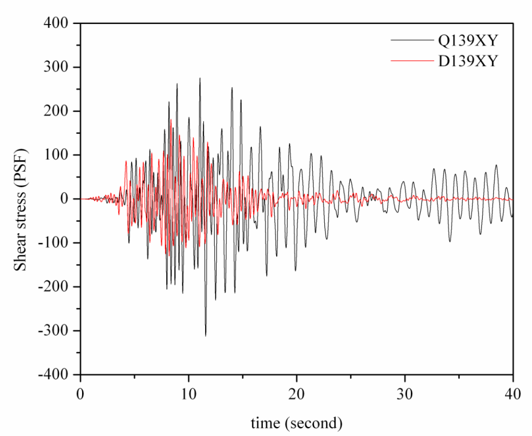

FIG. 12. Calculated shear stress history in 139th element.

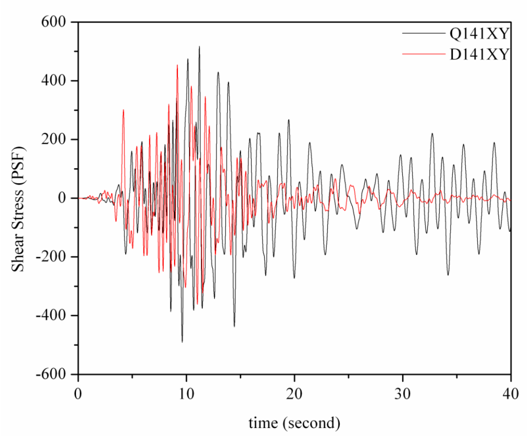

FIG. 13. Calculated shear stress history in 141st element.

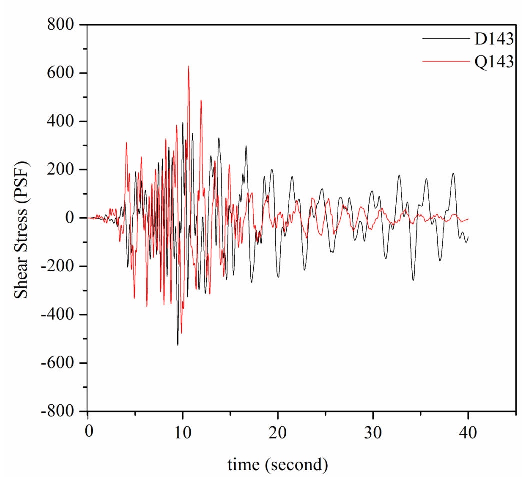

FIG. 14. Calculated shear stress history in 143 rd element.

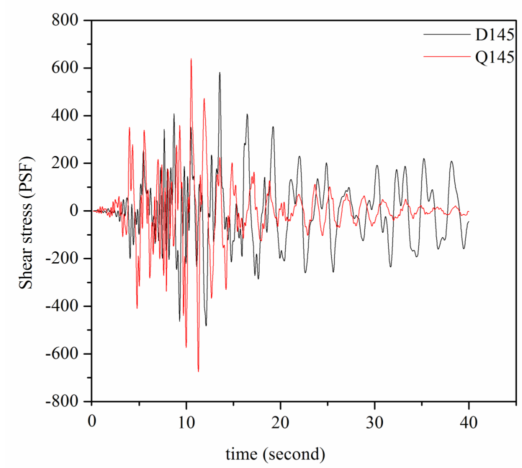

FIG. 15. Calculated shear stress history in 145th element.

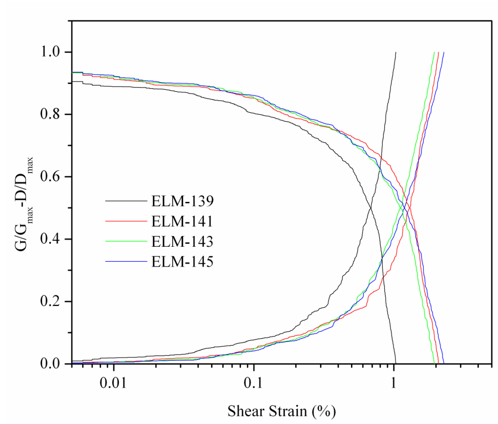

FIG. 16. Variation of shear modulus (G/G_{max}) and damping ratio (D/D_{max}) versus shear strain.

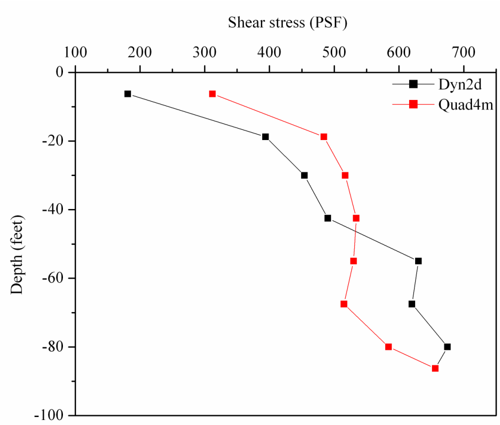

FIG. 17. Variation of maximum shear stresses with increasing depth.

CONCLUSION

The fractional time derivative scheme is used in Kelvin-Voigt type viscoelastic model in order to represent damping behavior in viscoelastic materials. This algorithm is compared with Rayleigh damping scheme. We consider that the damping matrix in Rayleigh scheme depends upon the arbitrarily determined parameters and thus has no physical meanings. However, the fractional-order damping scheme calculates the damping matrix directly from the strain history developed in the vibration system. The main disadvantages of fractional order damping scheme is the increasing numerical effort and the storages requirements due to local operators. But, the fractional order damping scheme permits for the continuous transition from the fluid to the solid state to represent damping behavior in soil materials.

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